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Asymptotic behavior of one-dimensional random dynamical systems — a revisit

Takehiko Morita[1]
Department of Mathematics,
Graduate School of Science, Osaka University

1. Introduction

We start with recalling the notion of nonsingular random dynamical system with stationary noise. Let $(X, \mathcal{B}, m)$ be a Lebesgue space (i.e. it is measurably isomorphic to the unit interval with the Lebesgue measure) and let $\{\tau_s\}_{s\in S}$ be a family of m-nonsingular transformations on $(X, \mathcal{B}, m)$ indexed by a polish space $(S, \mathcal{B}(S))$ such that the map $S\times X\ni(s, x)\mapsto\tau_{s}x\in X$ is $(\mathcal{B}(S)\times \mathcal{B})/\mathcal{B}$-measurable. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\sigma:\Omega\to\Omega$ a P-preserving transformation. Take an S-valued random variable ξ on $(\Omega, \mathcal{F}, P)$ and define an S-valued stationary process $\{\xi_{n}\}_{n=1}^{\infty}$ by $\xi_{n}=\xi\circ\sigma^{n-1}$ $(n\in \mathbb{N})$. We consider the family of random maps $X_{n}:X\to X$ given by

$$X_{0}(\omega)x=x,\ X_{n+1}(\omega)x=\tau_{\xi_{n+1}(\omega)}X_{n}(\omega)x \quad (n\geq 0) \tag{1.1}$$

for $(x, \omega)\in X\times\Omega$ and call it the random dynamical system with respect to $(\Omega, \mathcal{F}, P, \sigma, \{\tau_{s}\}_{s\in S}, \xi)$. As in Morita [7] (see also [5] and [8]), we introduce a skew product transformation $T : X\times\Omega\to X\times\Omega$ defined by

$$T(x, \omega)=(X_{1}(\omega)x, \sigma\omega) \quad \text{for } (x, \omega)\in X\times\Omega. \tag{1.2}$$

It is easy to see that T is an $(m\times P)$-nonsingular transformation. The relation between the asymptotic behavior of the random dynamical system X_{n} with respect to the reference measure m and the ergodic-theoretic properties of the skew product transformation T with restpect to $m\times P$ are studied in [5], [7], and [8].

The first aim of this article is to give some improvements of the results on one-dimensional random dynamical systems in the paper [7] obtained by the author. The other aim is to make remarks on S. R. B. measures and a sort of sample-wise central limit theorem for general cases.

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2. Ergodic properties of skew product transformations corresponding to one-dimensional random dynamical systems

In this section we consider the case when $X = [0,1]$ and each τ_s is a generalized Lasota-Yorke map (GLY map for short). An almost everywhere defined map $\tau : [0,1] \to [0,1]$ is called a GLT map if there exists a family $\mathcal{P}$ of closed intervals with nonempty interior and a family $\{\tau_J : J \in \mathcal{P}\}$ of maps of class C^2 satisfying the following.

(τ.1) $\mathrm{int} J \cap \mathrm{int} K = \emptyset$ for J, $K \in \mathcal{P}$ with $J \neq K$ and $m([0,1] \setminus \bigcup_{J \in \mathcal{P}} J) = 0$.

(τ,2)(regularity) $\tau_J|_{\mathrm{int}\, J} = \tau|_{\mathrm{int}\, J}$ for each $J \in \mathcal{P}$.

(τ.3)(finiteness) $\sharp\{K : K = \tau_J J\} < +\infty$.

(τ.4)(non-degeneracy) $d(\tau) = \inf_{J \in \mathcal{P}} \inf_{x \in J} |D\tau_J(x)| > 0$, where Df denotes the derivative of f.

(τ.5)(finite distorsion) $R(\tau) = \sup_{J \in \mathcal{P}} \sup_{x \in J} \dfrac{|D^2\tau_J(x)|}{(D\tau_J(x))^2} < +\infty$.

The quantities $d(\tau)$ and $R(\tau)$ are depending only on τ and independent of the choice of $\mathcal{P}$. For a GLY map τ, $\mathcal{P}(\tau)$ denotes the minimal family in the sense of refinement of measurable partition satisfying the conditions above. Note that the set of GLY maps is closed under composition. Put $\Delta(\tau) = \min\{m(\tau_J J) : J \in \mathcal{P}(\tau)\}$. We introduce the following.

$$\alpha(\tau) = d(\tau)^{-1}, \quad \beta(\tau) = 2\left(\frac{1}{\Delta(\tau)} + R(\tau)\right). \tag{2.1}$$

For the family $\{\tau_s\}_{s \in S}$ and for $N \in \mathbb{N}$ put

$$\alpha(s) = \alpha(\tau_s), \quad \beta_N(s_N, s_{N-1}, \ldots. s_1) = \beta(\tau_{s_N} \circ \tau_{s_{N-1}} \circ \cdots \circ \tau_{s_1}). \tag{2.2}$$

Consider the following conditions.

Condition A

$$M_0 = \sup\left\{ M \in [-\infty, +\infty] : \sum_{n=1}^{\infty} P\left(\frac{1}{n}\sum_{k=1}^{n} \log \alpha(\xi_k) \geq -M\right) < +\infty \right\}$$

>0.

Condition B There exists a positive integer $N > M_0^{-1}\log 2$ such that $\beta_N(\xi_N, \ldots, \xi_1)$ is integrable with respect to P, where M_0^{-1} is regarded as 0 if $M_0 = +\infty$.

REMARK 2.1. (1) Condition A is the same as the condition (A.1) in [7]. We mention about a few examples for which more intuitive conditions are sufficient for the validity of Condition A (see [7]).

(i) If there exists $d > 1$ satisfying $d(\tau_s) \geq d$ for all $s \in S$, then Condition A is valid.

(ii) If $\{\xi_n\}$ is a strongly mixing sequence with mixing coefficient $\{\phi_n\}$ satisfying

$$\sum_{n=1}^{\infty} n\phi(n) < \infty,$$

particularly, if $\{\xi_n\}$ is an independent and identically distributed sequence, then the condition

$$\int_{\Omega} \log \alpha(\xi)\, dP < 0 \tag{2.3}$$

is sufficient for Condition A. For the definition of strongly mixing property in this case see Chapter 18 in [3].

(iii) Let Ω be a compact commutative group and P the normalized Haar measure. We assume that there is an element $a \in \Omega$ such that $\{a^n\}_{n\in\mathbb{Z}}$ is dense in Ω. Let σ be the rotation on Ω given by a i.e. $\sigma\omega = a\omega$ for $\omega \in \Omega$. Let $S = \Omega$ and let ξ be the identity map. If there exists a continuous function φ on Ω satisfying $\alpha \leq \varphi$ and $\displaystyle\int_{\Omega} \log \varphi\, dP < 0$, we see that Condition A is satisfied.

(2) Condition B is much milder than the condition (A.2) in [7] which implies that

$$\sup_{s\in S} \alpha(s) < +\infty, \sup_{s\in S} \beta_1(s) < +\infty, \text{ and } \sup_{s_1,\ldots,s_N\in S^N} \beta_N(s_N, \ldots, s_1) < +\infty.$$

The following theorem is a renewal version of Theorem 2.1 in [7]. About the technical terms in ergodic theory in the below, weak-mixing, exactness, for example, the reader can consult [11].

THEOREM 2.2. *Let X_n be a random dynamical system with respect to $(\Omega, \mathcal{F}, P, \sigma, \{\tau_s\}_{s\in S})$ such that $\{\tau_s\}_{s\in S}$ is a family of GLY maps and let T be the corresponding skew product transformation. Assume that Condition A and Condition B are satisfied. Then we have the following.*

(1) *There exists an $(m\times P)$-absolutely continuous T-invariant probability measure.*

(2) (a) *If the measure-theoretic dynamical system (σ, P) is ergodic, there exists a finite number of $(m \times P)$-absolutely continuous T-invariant probability measures, say $Q_1, \dots, Q_r$ such that the measure-theoretic dynamical system (T, Q_i) is ergodic for each i $(1 \le i \le r)$ and any T-invariant $(m \times P)$-absolutely continuous probability measure can be expressed as a convex combination of Q_i's.*

(b) *For each i $(1 \le i \le r)$, we can find a finite number of disjoint measurable subsets $L_{i,0}, \dots, L_{i.N_i-1}$ of $[0,1] \times \Omega$ such that $TL_{i,j} = L_{i,j+1}$ $(\mathrm{mod}\ N_i)$ and if we put $Q_{i,j} = N_iQ_i|_{L_{i,j}}$ for j $(0 \le j \le N_i - 1)$, then it is T^{N_i}-invariant and the measure-theoretic dynamical system $(T^{N_i}, Q_{i,j})$ is totally ergodic.*

(c) *If the measure-theoretic dynamical system (σ, P) is weak-mixing, then so is the measure-theoretic dynamical system $(T^{N_i}, Q_{i,j})$ for each pair (i, j) with $1 \le i \le r$ and $0 \le j \le N_i - 1$.*

(d) *If the measure-theoretic dynamical system (σ, P) is exact, then so is the measure-theoretic dynamical system $(T^{N_i}, Q_{i,j})$ for each pair (i, j) with $1 \le i \le r$ and $0 \le j \le N_i - 1$.*

REMARK 2.3. In [7] the assertions (1) and (4) are proved provided that the assumptions (A.1) and (A.2) are fulfilled. The assertion (3) of Theorem 2.1 in [7] is corresponding to the assertion (b) in Theorem 2.2 in the above. The proof in [7] was carried out with the assumption of the total ergodicity of (σ, P). But now we have seen that the total ergodicity is too mach stronger than we need. The assertion (c) is novel.

Since we do not have enough space, we just restrict ourselves to state the basic lemma which plays important roles in proving Theorem 2.2. The detailed proof of the theorem will be given elsewhere.

For the sake of stating the basic lemma, we need the notion of Perron-Frobenius operators for nonsingular transformations. Let $(Y, \mathcal{C}, \nu)$ be a probability space and let $\tau : Y \to Y$ be a ν-nonsingular transformation. Then we can define a bounded linear operator $\mathcal{L}_{\tau,\nu} : L^1(\nu) \to L^1(\nu)$ characterized by the formula

$$\int_Y (\mathcal{L}_{\tau,\nu} f) g \, d\nu = \int_Y f(g \circ \tau) \, d\nu \quad \text{for } f \in L^1(\nu) \text{ and } g \in L^\infty(\nu).$$

One of the most important facts concerned with $\mathcal{L}_{\tau,\nu}$ is that for $h \in L^1(\nu)$, the complex-valued measure $h\nu$ with density h is τ-invariant if and only if h is fixed point of $\mathcal{L}_{\tau,\nu}$.

We also need the notion of total variation of Lebesgue measurable function on the interval. For a Lebesgue measurable function f on $[0,1]$, we put $\bigvee f = \inf \widetilde{\bigvee} \tilde{f}$, where the infimum is taken over all the versions of f and $\widetilde{\bigvee} \tilde{f}$ denotes the total variation of $\tilde{f}$.

In what follows we assume that Condition A and Condition B. Choose δ satisfying $2e^{-NM_0} < \delta < 1$, where M_0 and N are the numbers which appear in Condition A and Condition B, respectively. For any positive integers $p < n$, we put

$$\Omega_p^n = \bigcup_{j=p-1}^{n-1} \left(\alpha(\xi_n)\alpha(\xi_{n-1}) \cdot \cdots \cdot \alpha(\xi_{n-j}) \geq (2^{-1}\delta)^{(j+1)/N} \right).$$

Finally we introduce a function space $\mathbf{F} \subset L^\infty(m \times P)$. $\Phi \in L^\infty(m \times P)$ belongs to $\mathbf{F}$ if and only if for each ω, $\bigvee \Phi(\cdot, \omega) < \infty$ as a function on $[0,1]$ and the function $\bigvee \Phi$ is an element of $L^\infty(P)$, i.e. $\| \bigvee \Phi \|_{\infty,P} < \infty$.

Now we can state the basic lemma.

LEMMA 2.4. *Assume that Condition A and Condition B are valid. Then there exist a positive constant C and ρ with $0 < \rho < 1$ such that for any integer p, we can find $K_p \in L^1(P)$ such that for any integer $n > p$, function $\Phi \in \mathbf{F}$, and set $B \in \mathcal{B} \times \mathcal{F}$ we have*

$$\left| \int_B \mathcal{L}^n_{T,m\times P} \Phi \, d(m \times P) \right| \leq \int_{\Omega_p^n} \|\Phi\|_{1,m} \, dP + \int_B \mathcal{L}^n_{\sigma,P}(K_p \|\Phi\|_{1,m}) \, d(m \times P)$$
$$+ C\rho^n \| \bigvee \Phi \|_{\infty,P} (m \times P)(B).$$

We note that we make use of Lemma 2.4 in order to checking for $\Phi \in L^1(m \times P)$ how good the uniform integrability or weak compactness in $L^1(m \times P)$ of the sequence $\{\mathcal{L}^n \Phi\}_{n=0}^{\infty}$ is. For example, if we show that the set of $\Phi \in L^1(m \times P)$ for which the sequence $\{(1/n)\sum_{k=0}^{n-1} \mathcal{L}_{T,m\times P}^k \Phi\}_{n=1}^{\infty}$ is uniformly integrable is dense in $L^1(m \times P)$, then we see that the sequence $\{(1/n)\sum_{k=0}^{n-1} \mathcal{L}_{T,m\times P}^k\}_{n=0}^{\infty}$ of time-averaged Perron-Frobenius operators converges in the strong operator topology in $L^1(m \times P)$. This yields the validity of the assertion (1) in Theorem 2.2.To prove the basic lemma we need the following version of the Lasota-Yorke type inequality (Lemma 3.1 in [10], see also [6]).

LEMMA 2.5. *For any GLY map τ, we have*

$$\bigvee \mathcal{L}_{\tau,m} f \leq 2\alpha(\tau) \bigvee f + \beta(\tau)\|f\|_{1,m}.$$

REMARK 2.6. The assertion (1) in Theorem 2.2 is obtained as a corollary of much stronger result that for any $\Phi \in L^1(m \times P)$, the time average $\displaystyle\frac{1}{n}\sum_{k=0}^{n-1} \mathcal{L}_{T,m\times P}^k \Phi$ converges in $L^1(m \times P)$. This implies that for P-almost every ω, there exists $\Gamma(\omega) \in \mathcal{B}([0,1])$ with $m(\Gamma(\omega)) = 1$ such that for each $x \in \Gamma(\omega)$ we can find a Borel probability measure $\mu_{(x,\omega)}$ satisfying

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f(X_k(\omega)x) = \int_{[0,1]} f \, d\mu_{(x,\omega)}$$

for each $f \in C([0,1])$. In particular, the measure-theoretic dynamical system (σ, P) is ergodic, we can show that there is a positive integer r independent of ω such that there exist r Borel probability measures $\mu_{(1,\omega)}, \ldots, \mu_{(r,\omega)}$ on $[0,1]$ which are absolutely continuous with respect to the Lebesgue measaure m and a measurable partition $|\{\Gamma(1,\omega), \ldots, \Gamma(r,\omega)\}$ of $\Gamma(\omega)$ such that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f(X_k(\omega)x) = \int_{[0,1]} f \, d\mu_{(j,\omega)}$$

holds for $x \in \Gamma(j,\omega)$ and $f \in C([0,1])$. This sort of result can be found in Buzzi [1]. Each measure $\mu(j,\omega)$ is to be called a Sinai-Ruelle-Bowen measure (S. R. B. measure) or physical measure of the random dynamical system X_n. Moreover, in the case when $\{\xi_n\}$ is a independent sequence, the deterministic version lemma in [9] yields that the measures $\mu(j,\omega)$ and the sets $\Gamma(j,\omega)$ turn out to be independent of ω.

3. Remarks on physical measures and weak law of sample-wise central limit phenomena

In this section we develop the general theory of random dynamical systems. In what follows X_n is a random dynamical system whose state space X is a compact metric space. First we consider the situation as in Remark 2.6 in the previous section.

PROPOSITION 3.1. *Let X_n be a random dynamical system whose state space X is a compact metric space. Suppose that the sequence $\{(1/n)\sum_{k=0}^{n-1}\mathcal{L}_{T,m\times P}^k\}_{n=1}^{\infty}$ of time-averaged Perron-Frobenius operators converges to a projection with finite rank in the strong operator topology in $L^1(m\times P)$. Then, the noise dynamical system (σ,P) has a finite number of ergodic components, say $\Omega(1)$, ..., $\Omega(q)$. Consider an ergodic component $\Omega(i)$. Then there exists a positive integer r_i such that P-almost every $\omega \in \Omega(i)$, we can find a family $\{\Gamma(i,1,\omega), \ldots, \Gamma(i,r_i,\omega)\}$ of disjoint elements in $\mathcal{B}(X)$ with $m(\bigcup_{j=1}^{r_i}\Gamma(i,j,\omega)) = 1$ and a family $\{\mu_{(i,1,\omega)}, \ldots, \mu_{(i,r_i,\omega)}\}$ of m-absolutely continuous Borel probability measures such that*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} f(X_k(\omega)x) = \int_X f\,d\mu_{(i,j,\omega)}$$

holds for $x \in \Gamma(i,j,\omega)$ and $f \in C(X)$.

Sketch of Proof. Let r be the dimension of eigenspace of $\mathcal{L}_{T,m\times P}$ corresponding to the eigenvalue 1. Then the number of ergodic components of the $(m\times P)$-absolutely continuous T-invariant measure whose density is the strong limit of $(1/n)\sum_{k=0}^{n-1}\mathcal{L}_{T,m\times P}^k 1$ is r. Suppose that Λ_1, ..., Λ_p are disjoint σ-invariant elements in $\mathcal{F}$. It is easy to see that $X\times\Lambda_1$, ..., $X\times\Lambda_p$

are T-invariant. Thus p is not greater than r. This yields the finiteness of ergodic components of (σ, P).

Now we may assume that (σ, P) is ergodic. Let $H_1, \ldots, H_r$ be a family of density functions of ergodic invariant probability measures for T forming a basis of the eigenspace of $\mathcal{L}_{T,m\times P}$ corresponding to the eigenvalue 1. Put $E_i = (H_i > 0)$ for $i = 1, \ldots, r$. It is not so hard to show that there exists $F_i \subset E_i$ with $(m \times P)(F_i) = (m \times P)(E_i)$ such that for each $(x, \omega) \in C_i = \bigcup_{k=0}^{\infty} T^{-k} F_i$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(X_k(\omega)x) = \int_X f(x') \left(\int_\Omega H_i(x', \omega')\, P(d\omega') \right) m(dx')$$

holds for $f \in C(X)$ (cf. Section 6 in [8], see also Theorem VII.6.13 in [2]). On the other hand we can show that any invariant probability density H satisfies $\displaystyle\int_X H(x, \omega) m(dx) = 1$ for P-almost every ω by virtue of the ergodicity of (σ, P). Then by putting $\Delta_j = \{\omega \in \Omega : \displaystyle\int_X H_j(x, \omega) m(dx) = 1\}$ we have

$$P\left(\left\{ \omega \in \Omega \,:\, m\Big(\big(\bigcup_{j=1}^{r} C_j\big)_\omega\Big) = 1 \right\} \cap \bigcap_{k=1}^{r} \Delta_k \right) = 1.$$

Therefore it is easy to see that we obtain the desired result by putting $\Gamma(1, j, \omega) = (C_j)_\omega$ and $\mu(i, j) = h_j m$ for each j, where $h_j \in L^1(m)$ is defined by

$$h_j(x) = \int_\Omega H_j(x, \omega)\, P(d\omega)$$

and $(C_j)_\omega$ is the ω-section of C_j as usual.

□

We have explained about the case when a random dynamical system admits physical measures i.e. the strong law of large numbers holds with respect to the reference measure m for the system. Next we study the central limit theorem for random dynamical system with respect to the reference measure. For the sake of simplicity we assume that the noise (σ, P) is exact and the corresponding skew product transformation is also exact with respect to the unique $(m \times P)$-absolutely continuous invariant measure.

PROPOSITION 3.2. *Let X_n be a random dynamical system whose state space X is a compact metric space. Suppose that the sequence $\{\mathcal{L}^n_{T,m\times P}\}_{n=1}^{\infty}$ of iterated Perron-Frobenius operators converges in the strong operator topology in $L^1(m \times P)$ to the projection onto the one-dimensional space spanned by the unique invariant probability density H. Let g be a bounded real-valued function on X satisfying $\int_X g\, d\mu = 0$ for the unique physical measure μ. Consider the normalized partial sum*

$$S_n g(x,\omega) = \sum_{k=0}^{n-1} g(X_k(\omega)x) = \sum_{k=0}^{n-1} g(T^k(x,\omega)) \quad ((x,\omega) \in X \times \Omega).$$

Then the following are equivalent.

(1) *$S_n g/\sqrt{n}$ converges in law to the standard normal distribution with respect to the unique $(m \times P)$-absolutely continuous invariant probability measure $Q = H \cdot (m \times P)$.*

(2) *$S_n g/\sqrt{n}$ converges in law to the standard normal distribution with respect to $m \times P$.*

(3) *$S_n g/\sqrt{n}$ converges in law to the standard normal distribution with respect to any $(m \times P)$-absolutely continuous probability measure.*

(4) *For any continuous function u on $\mathbb{R}$ with compact support*

$$\int_X u\left(\frac{S_n g(x,\omega)}{\sqrt{n}}\right) m(dx) \to \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(t) e^{-t^2/2}\, dt$$

in probability with respect to P as $n \to \infty$.

(5) *For any continuous function u on $\mathbb{R}$ with compact support and m-absolutely continuous probability measure ν*

$$\int_X u\left(\frac{S_n g(x,\omega)}{\sqrt{n}}\right) \nu(dx) \to \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(t) e^{-t^2/2}\, dt$$

in probability with respect to P as $n \to \infty$.

(6) *For any continuous function u on $\mathbb{R}$ with compact support*

$$\int_X u\left(\frac{S_n g(x,\omega)}{\sqrt{n}}\right) m(dx) \to \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} u(t)e^{-t^2/2}\,dt$$

in $L^1(P)$ as $n \to \infty$.

(7) *For any continuous function u on $\mathbb{R}$ with compact support and m-absolutely continuous probability measure ν*

$$\int_X u\left(\frac{S_n g(x,\omega)}{\sqrt{n}}\right) \nu(dx) \to \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} u(t)e^{-t^2/2}\,dt$$

in $L^1(P)$ as $n \to \infty$.

Sketch of Proof. We just show how to get (7) from (1). We assume that the distribution of the normalized partial sum $S_n g/\sqrt{n}$ with respect to $Q = H\cdot(m\times P)$ converges in law to the standard norma distribution. Choose any real-valued element $u \in C_c(\mathbb{R})$, where $C_c(\mathbb{R})$ is the totality of continuous functions on $\mathbb{R}$ with compact support. In addition, we choose sequences $\{p_n\}$ and $\{q_n\}$ of positive integers such that $n = p_n + q_n$, $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n = +\infty$ and $\lim_{n\to\infty} q_n/n = 0$. Then we have for $\Phi \in L^1(m \times P)$

$$\begin{aligned}
&\limsup_{n\to\infty} \int_{X\times\Omega} \Phi\cdot u(S_n g/\sqrt{n})\,d(m\times P)\\
&= \limsup_{n\to\infty} \int_{X\times\Omega} \Phi\cdot u((S_{p_n}g)\circ T^{q_n}/\sqrt{n} + S_{q_n}g/\sqrt{n})\,d(m\times P)\\
&= \limsup_{n\to\infty} \int_{X\times\Omega} \Phi\cdot u((S_{p_n}g)\circ T^{q_n}/\sqrt{n})\,d(m\times P)\\
(3.1)\qquad &= \limsup_{n\to\infty} \int_{X\times\Omega} (\mathcal{L}^{q_n}_{T,m\times P}\Phi)\cdot u(\sqrt{(p_n/n)}S_{p_n}g/\sqrt{p_n})\,d(m\times P)\\
&= \lim_{n\to\infty} \int_{X\times\Omega} (\mathcal{L}^{q_n}_{T,m\times P}\Phi)\cdot u(S_{p_n}g/\sqrt{p_n})\,d(m\times P)\\
&= \lim_{n\to\infty} \int_{X\times\Omega} \Phi\,d(m\times P)\int_{X\times\Omega} u(S_{p_n}g/\sqrt{p_n})H\,d(m\times P)\\
&= \int_{X\times\Omega} \Phi\,d(m\times P)\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} u(t)e^{-t^2/2}\,dt
\end{aligned}$$

Note that we have used the convergence assumption on $\mathcal{L}^n_{T,m\times P}$ to obtain the sixth line from the fifth line in the above. Clearly if we replace 'lim sup' by 'lim inf' we have the same equation as (3.1). Therefore we have

$$\tag{3.2} \begin{aligned} &\lim_{n\to\infty}\int_{X\times\Omega} \Phi\cdot u(S_n g/\sqrt{n})\, d(m\times P) \\ &= \int_{X\times\Omega} \Phi\, d(m\times P)\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} u(t)e^{-t^2/2}\, dt. \end{aligned}$$

Taking $\Phi(x,\omega) = f(x)$ for $f \in L^1(m)$ with $\int_X f\, dm = 1$, we obtain

$$\tag{3.3} \lim_{n\to\infty}\int_\Omega \left(\int_X u(S_n g/\sqrt{n})\cdot f\, dm\right) dP = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} u(t)e^{-t^2/2}\, dtt.$$

Next we consider the probability measure ν on X with density f and $C_0(\mathbb{R})^*$-valued random variable φ_n satisfying for $u \in C_0(\mathbb{R})$

$$\langle \varphi_n(\omega), u\rangle = \int_X u(S_n g(x,\omega)/\sqrt{n}) f(x)\, m(dx),$$

where $C_0(\mathbb{R})$ is the Banach space obtained by the completion of $C_c(\mathbb{R})$ with respect to the supremum norm, i.e. the space of all continuous functions u on $\mathbb{R}$ with $\lim_{|t|\to\infty} u(t) = 0$. By virtue of Theorem V.4.2 (Alaoglu Theorem) and Theorem V.5.1 in [2], the closed unit ball of $C_0(\mathbb{R})^*$ is a compact metrizable space. Therefore $\{\varphi_n\}$ is a sequence of random variables taking values in a compact metrizable space. Thus it is tight. Take any subsequence converging in law. Then by (3.3) we can show that the limit φ is not random and satisfies

$$\langle \varphi(\omega), u\rangle = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} u(t)e^{-t^2/2}\, dt$$

for $u \in C_0(\mathbb{R})$. This yields that φ_n converges in law to $C_0(\mathbb{R})^*$-valued random variable which is constantly the standard norma distribution $N(0,1)$.

Now we define the function $F : C_0(\mathbb{R})^* \to \mathbb{C}$ by

$$F(\varphi) = \left| \langle \varphi, u\rangle - \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} u(t)e^{-t^2/2}\, dt \right|.$$

Obviously, it is continuous on the unit closed ball of $C_0(\mathbb{R})^*$. Thus we arrive at

$$\lim_{n\to\infty}\int_\Omega F(\varphi_n(\omega))\,P(d\omega)=\int_\Omega F(N(0,1))\,P(d\omega)=0.$$

Hence we have verified that (7) is valid.

□

REMARK 3.3. The central limit theorem for random dynamical system given by the random iteration of Lasota-Yorke maps with independent $\{\xi_n\}$ is discussed in Ishitani [4]. In [4] it is shown that under an appropriate condition the central limit theorem of mixed type holds with respect to the product measure $\nu\times P$, where ν is any probability measure being absolutely continuous with respect to the Lebesgue measure on the unit interval. But we can not find literatures which treat the sample-wise central limit phenomena. So Proposition 3.2 might have novelty. In this stage the author does not know whether it is possible to replace 'in probability' by 'almost surely' in the assertion (3) in Proposition 3.2.

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Department of Mathematics
Graduate School of Science
Osaka University
Toyonaka, Osaka 560-0043
JAPAN

大阪大学・大学院理学研究科　盛田 健彦